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On Garsia–Remmel problem of rook equivalence

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Abstract

In 1987, Garsia and Remmel proved a theorem that two Ferrers boards share the same rook polynomial if and only if they share all the Garsia–Remmel q-rook polynomials. In their paper they proposed the problem to find a purely combinatorial proof to this theorem. This note gives such a proof.

1. Introduction

Given any partition λ and any non-negative integer k, let R_λ^k denote the set of non-attacking rook placements involving k rooks on the Ferrers board F_λ, and let $R_k(\lambda; q)$ denote the Garsia–Remmel q-rook polynomial in the variable q as introduced in [3] (see Definition 5). Then Garsia and Remmel proved the following result.

Theorem 1. *For any given partitions λ and λ',*

$$|R_\lambda^k| = |R_{\lambda'}^k|, \quad \forall k \tag{1}$$

if and only if

$$R_k(\lambda, q) = R_k(\lambda', q), \quad \forall k\,. \tag{2}$$

In their paper, they asked for a purely combinatorial proof of this theorem. The purpose of this note is to give such a proof. Our idea is roughly as follows. In [2], Foata and Schützenberger produced an explicit bijection (i.e., the Foata–Schützenberger

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correspondence)

$$R^k_\lambda \longleftrightarrow R^k_{\lambda'}$$

for any partition λ and λ' satisfying (1) above. On the other hand, Ding [1] showed that there is a natural way to associate a 'length' $l(\sigma)$ to any $\sigma \in R^k_\lambda$. This gives rise to the partition

$$R^k_\lambda = \bigcup_{l \geqslant 0} R^k_{\lambda, l},$$

where

$$R^k_{\lambda, l} := \{\sigma \in R^k_\lambda \mid l(\sigma) = l\}.$$

The basis of our proof is the observation that the above mentioned Foata–Schützenberger correspondence leaves the length function invariant, giving in a bijection

$$R^k_{\lambda, l} \longleftrightarrow R^k_{\lambda', l} \quad \forall k, \ \forall l$$

for any partition λ, λ' satisfying (1). In particular, (1) holds if and only if

$$|R^k_{\lambda, l}| = |R^k_{\lambda', l}|, \quad \forall k, \ \forall l. \tag{3}$$

But it is already a consequence of a theorem in [1] (see Theorem 8 for the statement) that (3) holds if and only if (2) holds. Now Theorem 1 is immediate.

We organize this note in the following way: in Section 2, we give the notations and the necessary background of this problem. In Section 3, we describe the Foata–Schützenberger correspondence and an example. In Section 4, we give a combinatorial proof for Garsia–Remmel Theorem.

2. Notations and background

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition. Write $n = \lambda_1$. A Ferrers board F_λ is a two-dimensional subarray of an $m \times n$ matrix where the ith row has λ_i $*$'s right justified. For example, if $\lambda = (3, 1)$, then

$$F_\lambda = \begin{pmatrix} * & * & * \\ & & * \end{pmatrix}.$$

We recognize the right justification is not standard. The reason we do it is to include the parabolic boards as a special case. For details, see [1]. Let $M(F_\lambda)$ be the set of all $m \times n$ complex matrices (a_{ij}) such that $a_{ij} = 0$ for $(i, j) \notin F_\lambda$. Let $E_{ij} \in M(F_\lambda)$ be the matrix with 1 at (i, j) position and 0 elsewhere.

Let W_k be the symmetric group on k letters. Let

$$S(k) = \{(12), (23), \ldots, (k-1, k)\}$$

be the set of distinguished generators, which we sometimes view as $k \times k$ permutation matrices. A k-rook placement σ on F_λ is an assignment of k rooks on F_λ in non-attacking positions. The rook polynomial of the Ferrers board F_λ is defined by

$$f(\lambda, x) = \sum_{k=0}^{m} |R_\lambda^k| x^k .$$

In his book on combinatorial analysis [4], Riordan gave a survey on the early development of the theory of rook polynomials. In particular, there is a very interesting problem proposed there: if F_λ and $F_{\lambda'}$ are two Ferrers boards, under what condition we have $f(\lambda, x) = f(\lambda', x)$? In general, if two Ferrers boards F_λ and $F_{\lambda'}$ share the same rook polynomial, they are called rook equivalent boards. In this case, we write $F_\lambda \sim F_{\lambda'}$, i.e.,

$$F_\lambda \sim F_{\lambda'}$$

if and only if

$$f(\lambda, x) = f(\lambda', x)$$

if and only if

$$|R_\lambda^k| = |R_{\lambda'}^k|, \quad \forall k.$$

In 1970, Foata and Schützenberger introduced a very powerful construction which was called '(k, k')-transform' [2] and we will call these transforms Foata–Schützenberger correspondences. By means of these correspondences, Foata and Schützenberger classified Ferrers boards according to rook polynomials as follows.

Theorem 2. *Every Ferrers board is rook equivalent to a unique increasing Ferrers board. Here, a Ferrers board is said to be increasing if the heights of the columns from left to right increase strictly.*

In 1986, Garsia and Remmel considered a q-analogue of the rook number $|R_\lambda^k|$ [3]. First, they introduced a numerical function on R_λ^k which is called the Garsia–Remmel function in [1].

Definition 3. For each $\sigma \in R_\lambda^k$, put a dot in every cell of F_λ which is above or to the right of a rook and put a circle in each of the remaining cells of F_λ. Let $\mathrm{GR}(\sigma)$ be the number of circles.

Example 4. Suppose $\lambda = (4,3,2)$ and $\sigma = E_{12} + E_{33}$. Then, we get the following configuration:

$$\begin{pmatrix} 0 & 1 & \bullet & \bullet \\ & 0 & \bullet & 0 \\ & & 1 & \bullet \end{pmatrix}.$$

Thus, $\mathrm{GR}(\sigma) = 3$. Usually, we use a 1 to represent a rook as shown above.

Definition 5. The Garsia–Remmel q-rook polynomials are defined by

$$R_k(\lambda, q) = \sum_{\sigma \in R_\lambda^k} q^{\mathrm{GR}(\sigma)}, \quad k \geqslant 0.$$

where $\mathrm{GR}(\sigma)$ is as in Definition 3.

Remark. In [3], the authors called these polynomials the q-rook numbers. Clearly, this is a q-analogue of the rook number $|R_\lambda^k|$ because $R_k(\lambda, 1) = |R_\lambda^k|$. In [1], Ding introduced another numerical function on R_λ^k, called the *length* function. Roughly, the length function counts the minimun row and/or column adjacent transpositions to get σ from $v_r = \sum_{r=1}^k E_{i,n-k+i}$. For example, if $\lambda = (3,3,1)$ and $r = 2$, then

$$v_r = \begin{array}{|c|c|c|} \hline & 1 & \\ \hline & & 1 \\ \hline \multicolumn{2}{c|}{} & \\ \cline{3-3} \end{array}$$

Take

$$\sigma = \begin{array}{|c|c|c|} \hline & 1 & \\ \hline 1 & & \\ \hline \multicolumn{2}{c|}{} & \\ \cline{3-3} \end{array}$$

Then, it is easy to check the minimum number of adjacent row and/or column transpositions to get σ from v is 3. Formally, we have the following definition.

Definition 6. Let $v = \sum_{r=1}^k E_{i,n-k+i}$. For $\sigma \in R_\lambda^k$, the length function $l(\sigma)$ is defined by

$$l(\sigma) = \min\{k + h \mid \sigma = s_k \cdots s_1 v_r s_1' \cdots s_h'\},$$

where $s_i \in S(m)$ and $s'_j \in S(n)$ and

$$s_p \cdots s_1 v_r s'_1 \cdots s'_q \in R^r_\lambda$$

for each $p \in [k]$ and $q \in [h]$.

Definition 7. The rook length polynomials are defined by

$$\mathrm{RL}_k(\lambda, q) = \sum_{\sigma \in R^k_\lambda} q^{l(\sigma)}, \quad k \geqslant 0.$$

Clearly, this is another q-analogue of $|R^k_\lambda|$. It was shown in [1] that there is a very simple relation between Garsia–Remmel polynomials and rook length polynomials.

Theorem 8. *If* $\sigma \in R^k_\lambda$ *then*

$$\mathrm{GR}(\sigma) + l(\sigma) = C, \qquad R_k(\lambda, q) = q^C \mathrm{RL}_k(\lambda, q^{-1}),$$

where

$$C = \sum_{i=1}^{m} \lambda_i - \frac{k(k+1)}{2}.$$

Definition 9. Two Ferrers boards F_λ and $F_{\lambda'}$ are said to be q-rook equivalent if

$$R_k(\lambda, q) = R_k(\lambda', q), \quad k \geqslant 0.$$

In this case, we write $F_\lambda \sim_q F_{\lambda'}$. Thus the previous theorem tells us that $F_\lambda \sim_q F_{\lambda'}$ if and only if

$$\mathrm{RL}_k(\lambda, q) = \mathrm{RL}_k(\lambda', q), \quad k \geqslant 0.$$

Hence the Garsia–Remmel (Theorem 1) can be restated as follows.

Theorem 10. $F_\lambda \sim F_{\lambda'}$ *if and only if* $F_\lambda \sim_q F_{\lambda'}$.

We will provide a combinatorial proof for the theorem in Section 4.

3. Foata–Schützenberger correspondence

Let F_λ be a Ferrers board. The (i, j) cell in the board is said to be *admissible* if

$$0 \leqslant \lambda^{\mathrm{T}}_j - i \leqslant \lambda_i - j \leqslant \lambda^{\mathrm{T}}_{j+1} - i. \tag{4}$$

Then, for an admissible cell (i, j), the Foata–Schützenberger correspondence is as follows.

(a) Replace the subboard with the (i, j) cell as the northeast corner by its transpose. Eq. (4) guarantees that the board obtained in this way is again a Ferrers board. Write the new Ferrers board as $F_{\lambda(i,j)}$.

(b) Let σ be a rook placement on F_λ. For a given admissible cell (i, j), cut the Ferrers board into four pieces, say A, B, C and D. Accordingly, we cut $F_{\lambda(i,j)}$ into four pieces A', B', C' and D' (see the diagrams at the end of this section).

Consider the situation in σ. First, cross out all the columns in F_λ in which there is a rook in the C part. Then, cross out all the rows in F_λ in which there is a rook in the C part. We call the remaining rows and columns the residue rows and residue columns. For the fixed rook placement σ on F_λ, we construct another rook placement σ' on $F_{\lambda(i,j)}$.

(1) The rooks in B' are in the same positions as those in B.

(2) The restriction of σ' on C' is the transpose of the restriction of σ on C.

Thus, we can get the residue rows and columns of $F_{\lambda(i,j)}$ according to $\sigma'|_{C'}$.

(3) Put a rook in the sth row and the tth residue column from right in A' if and only if there is a rook in the sth row and the tth residue column from right in A.

(4) Put a rook in the sth residue row and the tth column in D' if and only if there is a rook in the sth residue row and the tth column in D.

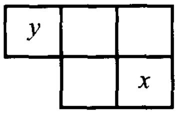
The construction above is called a Foata–Schützenberger correspondence, denoted as $\Phi(i, j)$. Clearly, this is a bijective involution from R_λ^k and $R_{\lambda(i,j)}^k$ if (i, j) is admissible.

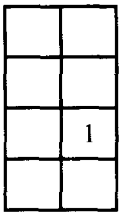
The following is an example. In order to single out the rooks in the C part, we denote them as x and y, instead of 1's.

$$\sigma = \begin{array}{|c|c|c|c|c|}\hline & & & 1 & \\ \hline & 1 & & & \\ \hline y & & & & \\ \hline \multicolumn{1}{c|}{} & & x & & \\ \cline{2-5} \multicolumn{3}{c|}{} & & 1 \\ \cline{4-5} \multicolumn{3}{c|}{} & & \\ \cline{4-5}\end{array}$$

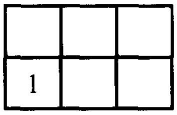
In this Ferrers board, the cell $(3,3)$ is admissible. Cut the board together with the rooks in it into four pieces.

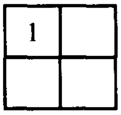
$$A = \begin{array}{|c|c|c|}\hline & & \\ \hline & 1 & \\ \hline \end{array} \qquad B = \begin{array}{|c|c|}\hline 1 & \\ \hline & \\ \hline \end{array}$$

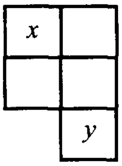
$C =$ 

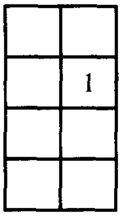
$D =$ 

Replace the C by its transpose $C' = C^{\mathrm{T}}$ and rearrange the rooks according to the rules given above, we have $\Phi(3,3)(\sigma)$ consisting of four pieces as follows.

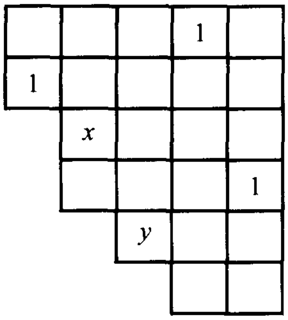
$A' =$ 

$B' =$ 

$C' =$ 

$D' =$ 

Put these four pieces together, we have

$\Phi(3,3)(\sigma) =$ 

4. Proof of Theorem 10

In [1], there is a very simple formula to evaluate the length function l.

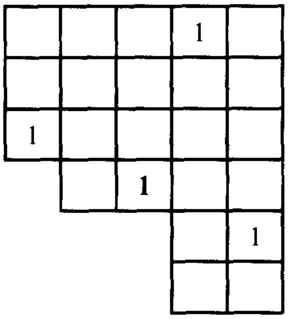
Theorem 11. *Let $\sigma \in R^r_\lambda$ such that $\sigma = \sum_{i=1}^{r} E_{x_i,y_i}$ and $x_1 < x_2 < \cdots < x_r$. We refer to the rook at (x_i, y_i) position as the ith* 1. *Let α_i be the number of zero rows above* *the x_ith row in σ, γ_i the number of zero columns to the right of the y_ith column in σ, and β_i the number of* 1*'s to the 'northeast' of the ith* 1. *Then,*

$$l(\sigma) = \sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i).$$

We call the formula above the local formula of length function l since it counts the contribution of each rook to the length function, individually. Recall that we use the residue addresses in the A part. The number of zero columns to the right of a rook in A part equals the number of zero columns to the right of the corresponding rook in A' part. Similarly, the number of zero rows above a rook in D part equals the number of zero rows above the corresponding rook in D' part. Clearly, the number of zero rows above a rook in A (the number of zero columns to the right of a rook in D) is the number of the corresponding rook in $A'(D')$. Moreover, $B = B'$. So we need only consider the contribution of the rooks in C and those in C'.

In fact, the local formula suggests a method to count the contribution of each rook to the length function by a confinguration in the hook of this rook. To illustrate our idea, we give the following example.

Example 12. Let $\lambda = (5,5,5,4,2,2)$. Let $\sigma \in R_\lambda^k$. $\sigma = E_{14} + E_{31} + E_{43} + E_{55}$. Then, σ is shown as in the following diagram.



Consider the contribution of the rook in bold face (we call this rook the current rook) to the length function $l(\sigma)$. Call the set of cells above or to the right of the $(4,3)$ cell the **hook** of the $(4,3)$ cell. If in any row above the current rook, there is a rook to the west of the current rook, then we put an X in the cell in the hook which is in that row. If there is a rook to the northeast or the southeast of the current rook, put an X in the cell in the hook which is in that colunm. Then, put an o in each of the empty box of the hook. Therefore, the local formula of the length function tells us that the number of o's in the hook is the contribution of the current rook as shown below.

		o	1	
		o		
1		X		
		1	X	X
				1

Note that the cell $(3,3)$ is admissible, $\Phi(3,3)(\sigma)$ is given in the following diagram.

			1	
	1			
				1
		1		

Put the X's in it, we have $\sigma' = \Phi(3,3)(\sigma)$ as the following.

	o		1	
	o			
	1	X	X	X
				1
		1		

Clearly, the number of o's in the hook is the same as that in σ.

In general, we have the following observations.

- The size of a hook is the same as its transpose.
- Under the Foata–Schützenberger correspondence, the rooks in the A and B parts, stay in the same rows in A' and B' parts.
- In C the rooks to the northeast of the current rook stay in C' to the northeast of the current rook there.
- In C, the rooks to the northwest of the current rook are transformed to the southeast of the current rook in C^{T}. Furthermore the rooks to the southeast of the current rook are transformed to the northwest of the current rook in C^{T}.

Therefore, by the local formula of the length function, $l(\sigma) = l(\Phi(i, j)(\sigma))$ if (i, j) is admissible.

References

[1] K. Ding, Invisible permutations and rook placements on a Ferrers board, in: Proc. 4th Internat. Conf. of Formal Power Series and Algebraic Combinatorics, Montreal, Quebec, Canada, 1992, 137–154; a journal version is to appear in: Discrete Math.

[2] D. Foata and M. Schützenberger, On the rook polynomials of Ferrers relations, in: P. Erdős et al., eds., Combinatorial Theory and its Applications, Vol. 2, Colloq. Math. Soc. János Bolyai, 4. (North-Holland, Amsterdam, 1970) 413–436.

[3] A.M. Garsia and J.B. Remmel, q-counting rook configurations and a formula of Frobenius, J. Combin. Theory Ser. A 41 (1986) 246–275.

[4] J. Riordan, An Introduction to Combinatorial Analysis (Wiley, New York, 1958) Ch. 7.